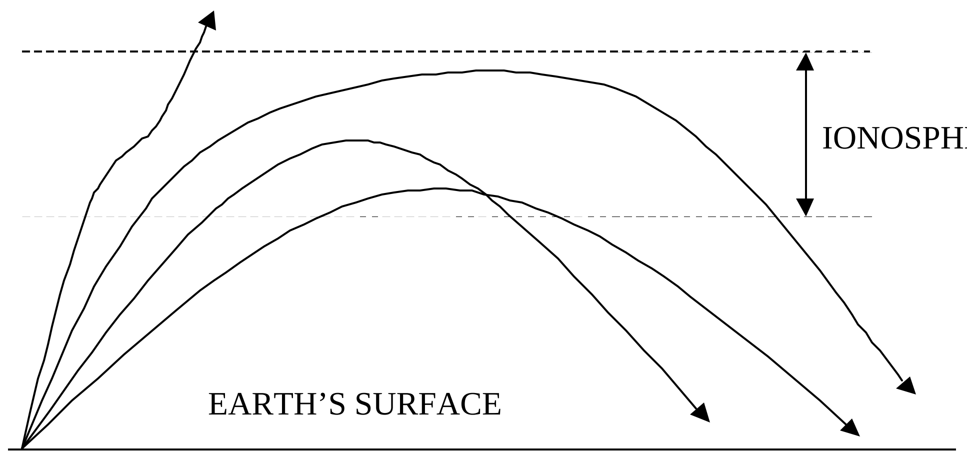
Naval Postgraduate School *Department of Electrical & Computer Engineering* *Monterey, California*

EC3630 Radiowave Propagation

IONOSPHERIC WAVE PROPAGATION

by Professor David Jenn

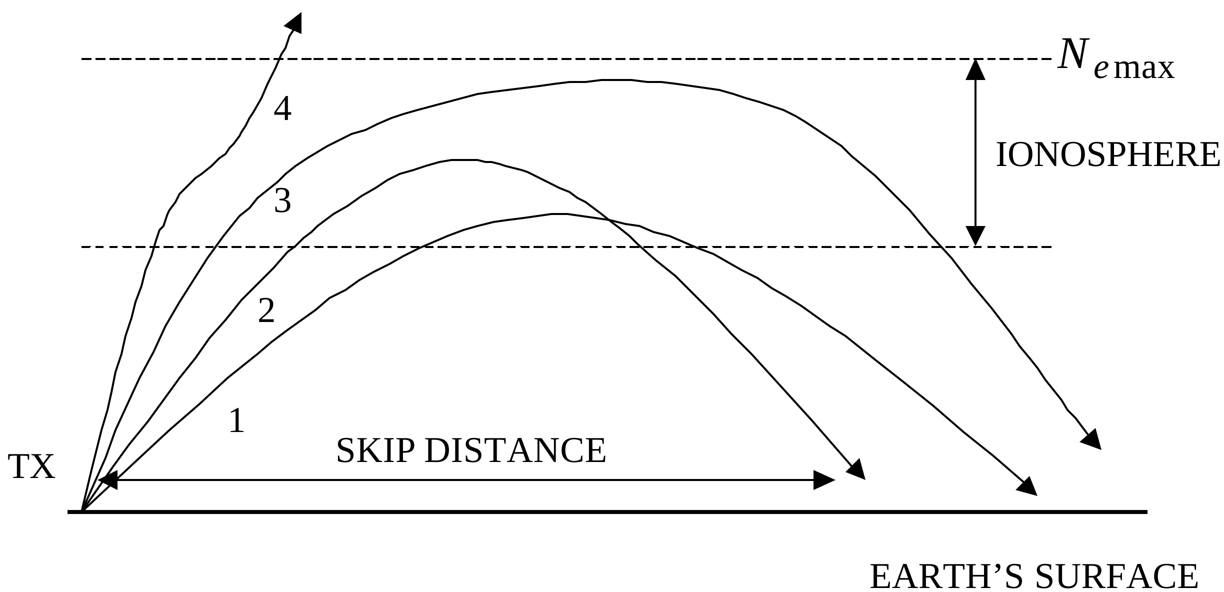


(version 1.4)

Ionospheric Radiowave Propagation (1)

The ionosphere refers to the upper regions of the atmosphere (90 to 1000 km). This region is highly ionized, that is, it has a high density of free electrons (negative charges) and positively charged ions. The charges have several important effects on EM propagation:

1. Variations in the electron density (N_e) cause waves to bend back towards Earth, but only if specific frequency and angle criteria are satisfied. Some examples are shown below. Multiple skips are common thereby making global communication possible.



Ionospheric Radiowave Propagation (2)

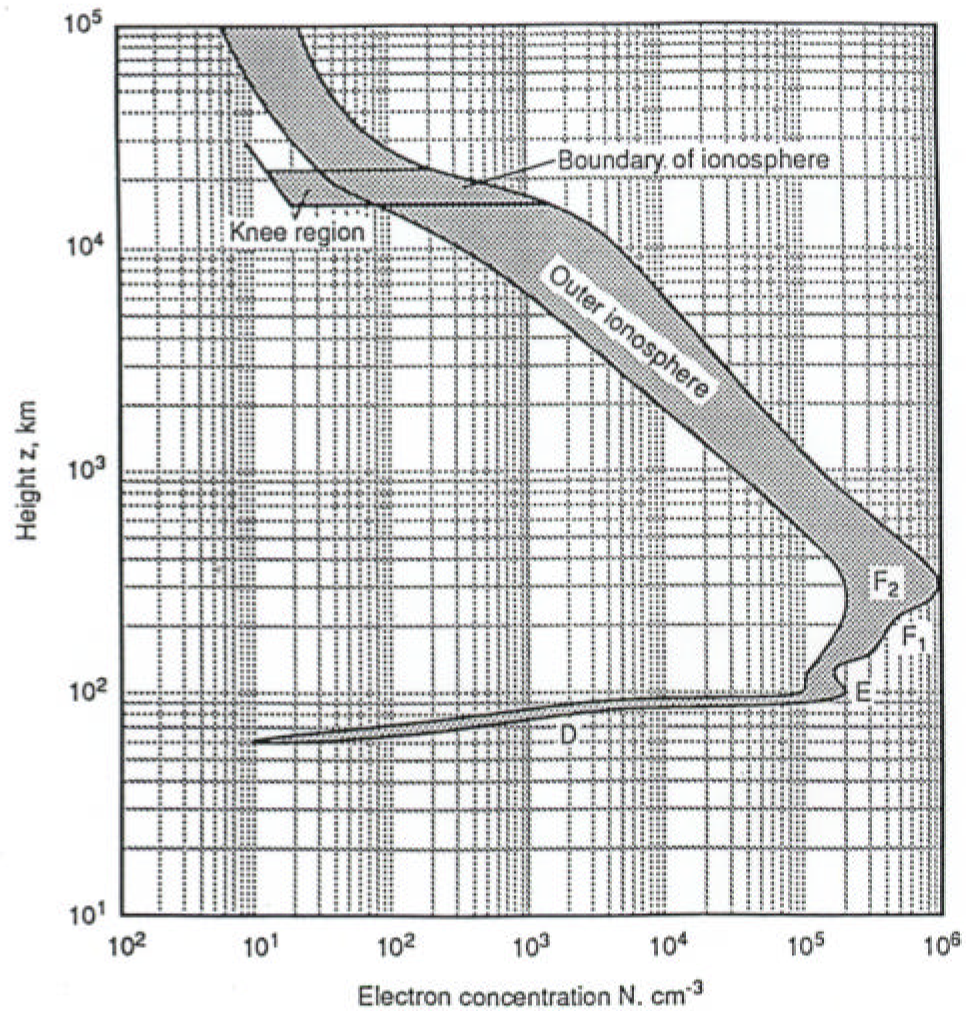
2. The Earth's magnetic field causes the ionosphere to behave like an anisotropic medium. Wave propagation is characterized by two polarizations (denoted as "ordinary" and "extraordinary" waves). The propagation constants of the two waves are different. An arbitrarily polarized wave can be decomposed into these two polarizations upon entering the ionosphere and recombined on exiting. The recombined wave polarization angle will be different that the incident wave polarization angle. This effect is called Faraday rotation.

The electron density distribution has the general characteristics shown on the next page. The detailed features vary with

- location on Earth,
- time of day,
- time of year, and
- sunspot activity.

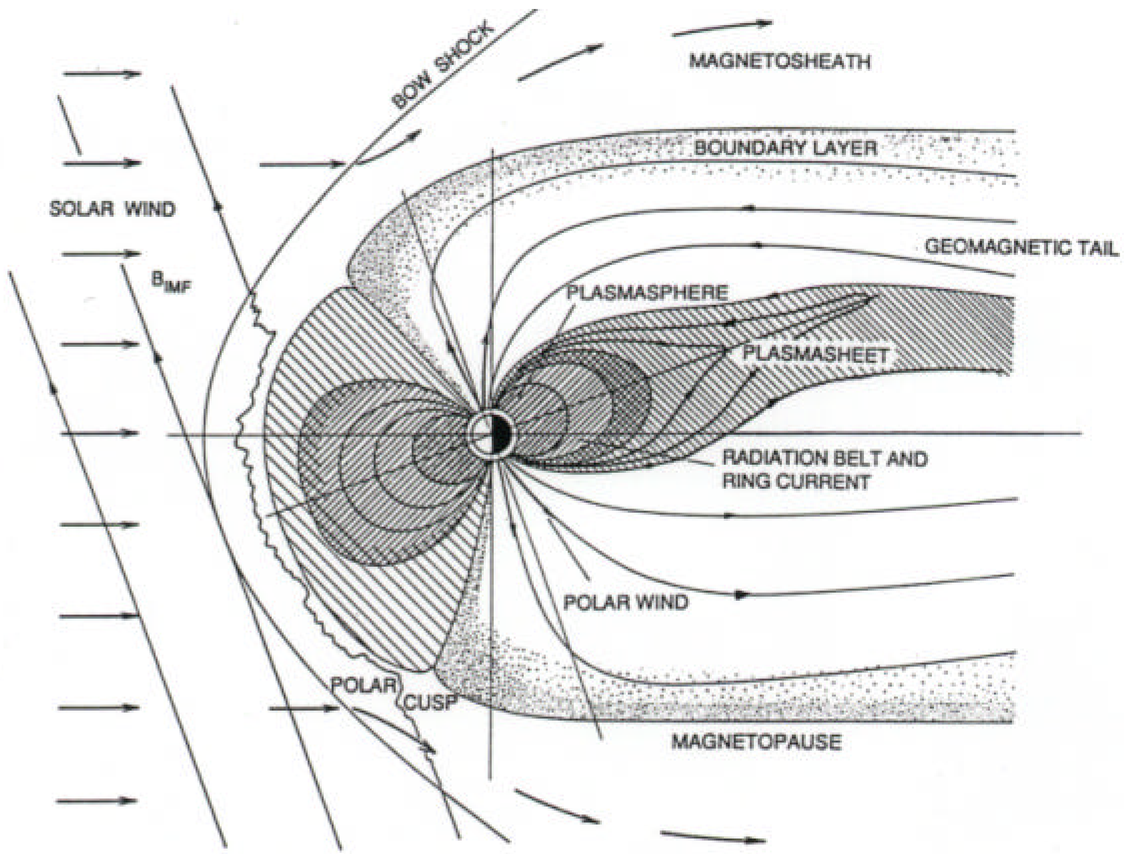
The regions around peaks in the density are referred to as layers. The F layer often splits into the F_1 and F_2 layers.

Electron Density of the Ionosphere



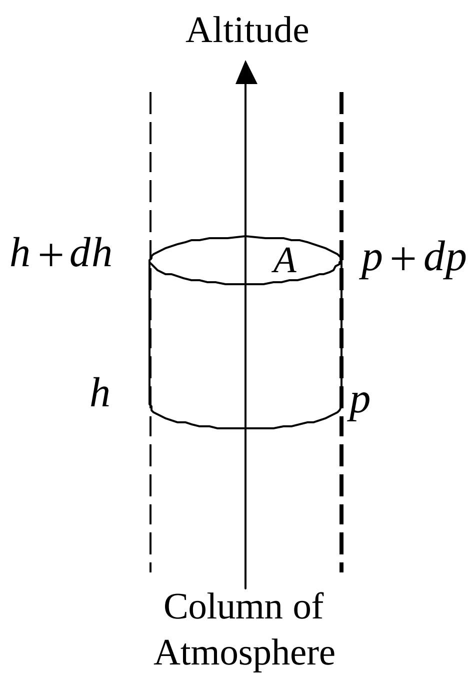
(Note unit is per cubic centimeter.)

The Earth’s Magnetosphere



Barometric Law (1)

Ionized layers are generated by solar radiation, mainly ultraviolet, which interacts with neutral molecules to produce electron-ion pairs. As a first step in predicting the ionized layers it is necessary to know the distribution of molecules with height.



Consider a differential volume of air, as shown in the figure. The pressure p decreases with altitude, so dp is negative. There is a net buoyant force on the volume of cross sectional area A

$$F_b = Ap - A(p+dp) = -Adp\,.$$

This force must be balanced by the gravitational force of the weight of the gas in the volume

$$F_g = rgdV = rgAdh$$

where r is the mass density of the gas (mass per unit volume) and g is the gravitational constant. If $\overline{m}$ is the mean molecular mass

$$F_g = rgdV = N\overline{m}gAdh$$

Equating the last two equations gives $-dp = N\overline{m}gdh$. N is the number of molecules per unit volume.

Barometric Law (2)

Pressure and temperature are related by the ideal gas law: $p = Nk\mathrm{T}$ (k is Boltzman's constant and T is temperature). Use the ideal gas law to substitute for N in dp:

$$-\frac{dp}{p} = -\frac{\overline{m}g}{k\mathrm{T}}dh$$

then integrate from starting height h_o to a final height h

$$\ln\frac{p(h)}{p(h_o)} = \int_{h_o}^{h}\frac{\overline{m}g}{k\mathrm{T}}dh'$$

Finally, assuming that all of the quantities are approximately constant over the heights of concern, and defining a new constant $\frac{1}{H} = \frac{\overline{m}g}{k\mathrm{T}}$, gives the barometric law for pressure vs. height:

$$p(h) = p(h_o)\exp\left(\int_{h_o}^{h}\frac{\overline{m}g}{k\mathrm{T}}dh'\right) = p(h_o)\exp\left(\int_{h_o}^{h}\frac{dh'}{H}\right) = p(h_o)\exp\left(-\frac{h-h_o}{H}\right)$$

H is called the local scale height and thus the exponent, when normalized by H, is in "H units." The reference height is arbitrary. The pressure, mass density, and number density are seen to vary exponentially with height difference measured in H units.

Chapman's Theory (1)

Chapman was the first to quantify the formation of ionized regions due to flux from the sun. In general we expect ion production to vary as follows:

1. Small at high altitudes – solar flux is high, but the number of molecules available for ionization is small
2. Small at low altitudes – the number of molecules available is large, but solar flux is low due to attenuation at higher altitudes
3. High at intermediate altitudes – sufficient number of molecules and flux

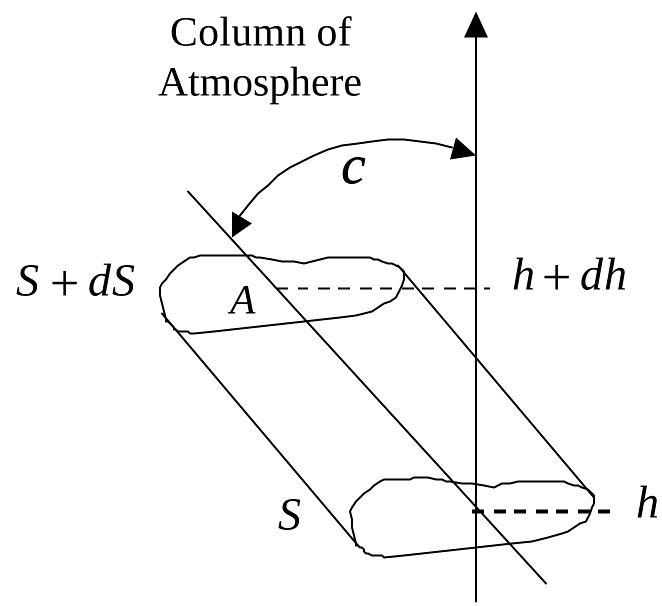
Chapman's derivation proceeds along the following steps:

1. Find the amount of flux penetrating to an altitude h.
2. By differentiation, find the decrease in flux with incremental height dh.
3. The decrease in flux represents absorbed energy. From the result of step 2 an expression for ion production per unit volume at height h is derived.
4. The barometric law is used to find the optical depth at height h (optical depth is a measure of the opacity of the atmosphere above h).
5. The maximum height of electron-ion production is set as the reference height.
6. Assuming equilibrium (no vertical wind) laws of the variation of the electron density are derived based on the rates of electron production and loss.
7. A scaling law is developed so that ionization can be represented by a single variable rather than two (altitude and solar zenith angle).

Chapman's Theory (2)

Only the final results are presented here. S denotes the solar flux and c is the zenith angle. The electron density at height h and zenith angle c is given by

$$N_e(c,h) = N_o \exp\left[\frac{1}{2}\left(1 - z - \sec c \; e^{-z}\right)\right]$$



where $z = (h - h_o)/H$ and N_o is the electron density a the reference height h_o. This equation specifies a recombination type of layer (as opposed to an attachment type layer[1]). It can be rewritten (by a substitution of variables and rescaling) as

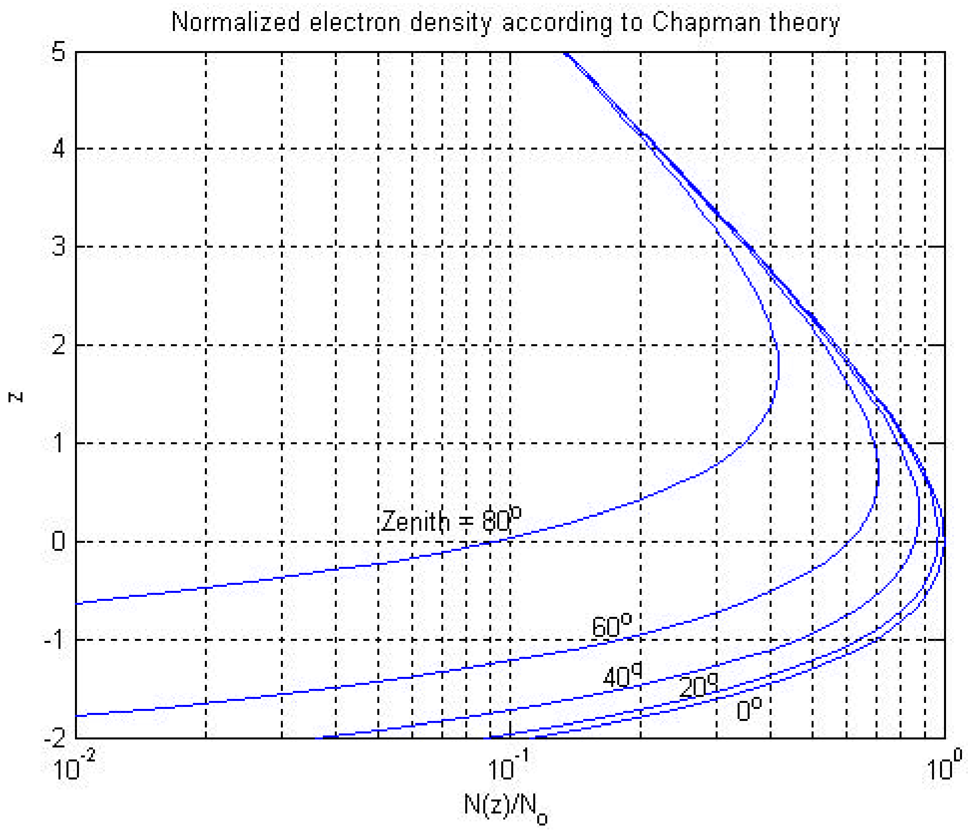
$$N_e(c,z) = \sqrt{\cos c}\; N_e(0, z - \ln \sec c)$$

The value for any c can be obtained from the $c = 0^\circ$ curve.

[1]There are several reactions which can remove electrons from the ionosphere. The two most important classes of recombination are (1) the electron attaches itself to a positive ion to form a neutral molecule, and (2) it attaches itself to a neutral molecule to form a negative ion.

Chapman's Theory (3)

It is found that the E and F1 regions behave closely to what is predicted by Chapman's equation.



Dielectric Constant of a Plasma (1)

When an electric field is applied, charges in the ionosphere are accelerated. The mass of the ions (positive charges) is much greater than the electrons, and therefore the motion of the ions can be neglected in comparison with that of the electrons. The polarization vector is

$$\vec{P} = -N_e \vec{p} = -N_e e \vec{d}$$

where N_e = electron density $/\,\mathrm{m}^3$, $e = 1.59 \times 10^{-19}$ C, electron charge, and $\vec{d}$ is the average displacement vector between the positive and negative charges. The relative dielectric constant is

$$e_r = \frac{\vec{D}}{e_o \vec{E}} = \frac{e_o \vec{E} + \vec{P}}{e_o \vec{E}} = 1 + \frac{\vec{P}}{e_o \vec{E}}$$

The propagation constant is

$$g = jk_c = j\frac{w}{c}\sqrt{e_r} = jk_o n$$

where $n = \sqrt{e_r}$ is the <u>index of refraction</u>. Both the dielectric constant and index of refraction can be complex ($e_r = e_r' - je_r''$ and $n = n' - jn''$).

Dielectric Constant of a Plasma (2)

Assume a x polarized electric field. The electrons must follow the electric field, and therefore, $\vec{d} = \hat{x}x$. The equation of motion of an electron is

$$m\ddot{x} + nm\dot{x} = -eE_x e^{jwt}$$

where $m = 9.0 \times 10^{-31}$ kg, electron mass, and n = collision frequency. The solution of the equation of motion is of the form $x = ae^{jwt}$. Substituting this back into the equation gives

$$m(-aw^2 e^{jwt}) + nm(jwae^{jwt}) = -eE_x e^{jwt}$$

with $a = \dfrac{eE_x}{mw^2(1 - jn/w)}$. The polarization vector is

$$\vec{P} = -\frac{N_e e^2 E_x e^{jwt}\hat{x}}{mw^2(1 - jn/w)}$$

Now define a <u>plasma frequency</u>, $w_p = \sqrt{\dfrac{N_e e^2}{me_o}}$ and constants $X = \left(\dfrac{w_p}{w}\right)^2$ and $Z = \dfrac{n}{w}$ so that

$$\vec{P} = -e_o E_x e^{jwt}\hat{x}\frac{X}{(1 - jZ)}.$$

Dielectric Constant of a Plasma (3)

The ratio of $\vec{P}$ to $\vec{E}$ gives the complex dielectric constant, which is equal to the square of the index of refraction

$$e_r = e_r' - j e_r'' = n^2 = 1 - \frac{X}{(1 - jZ)} = 1 - \frac{w_p^2}{w(w - jn)}$$

Separating into real and imaginary terms gives an equivalent conductivity $e_r = e_r' - j\dfrac{s}{we_o}$

where

$$e_r' = 1 - \frac{N_e e^2}{e_o m(n^2 + w^2)} \quad \text{and} \quad s = \frac{N_e e^2 n}{m(n^2 + w^2)}$$

For the special case of no collisions, $n = 0$, and the corresponding propagation constant is real

$$k_c = w\sqrt{m_o e_r e_o} = k_o\sqrt{1 - \frac{w_p^2}{w^2}}$$

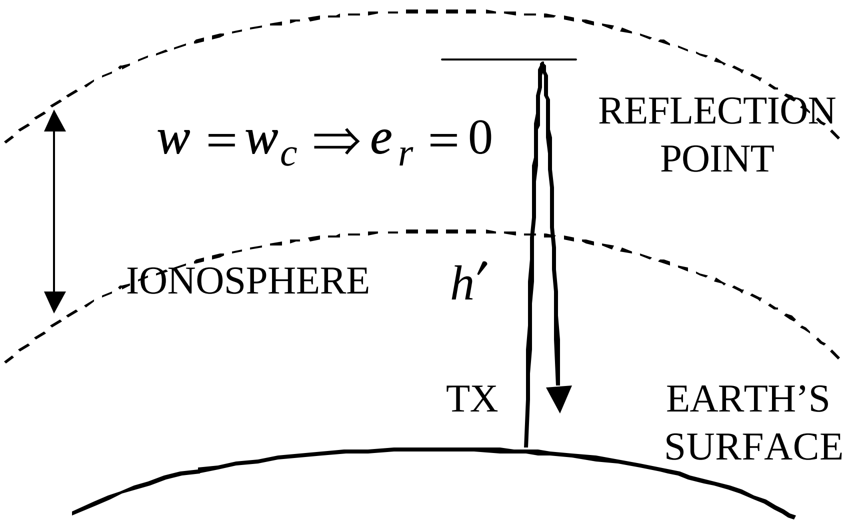
with $k_o = w\sqrt{m_o e_o}$.

Dielectric Constant of a Plasma (4)

Consider three cases:

1. $w > w_p$: k_c is real and $e^{-jk_c z} = e^{-j|k_c|z}$ is a propagating wave
2. $w < w_p$: k_c is imaginary and $e^{-jk_c z} = e^{-|k_c|z}$ is an evanescent wave
3. $w = w_p$: $k_c = 0$ and this value of w is called the critical frequency[1], w_c

At the critical frequency the wave is reflected. Note that w_c depends on altitude because the electron density is a function of altitude. For electrons, the highest frequency at which a reflection occurs is



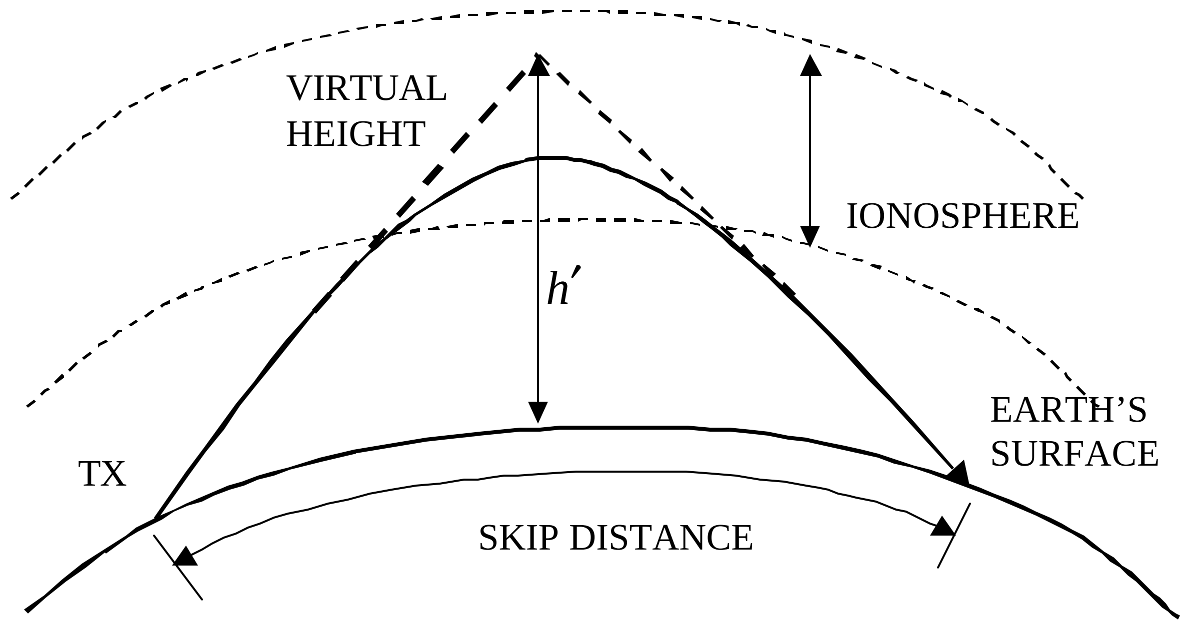
$$f_c = \frac{w_c}{2p} \approx 9\sqrt{N_{e\,\max}}$$

Reflection at normal incidence requires the greatest N_e.

[1] The critical frequency is where the propagation constant is zero. Neglecting the Earth's magnetic field, this occurs at the plasma frequency, and hence the two terms are often used interchangeably.

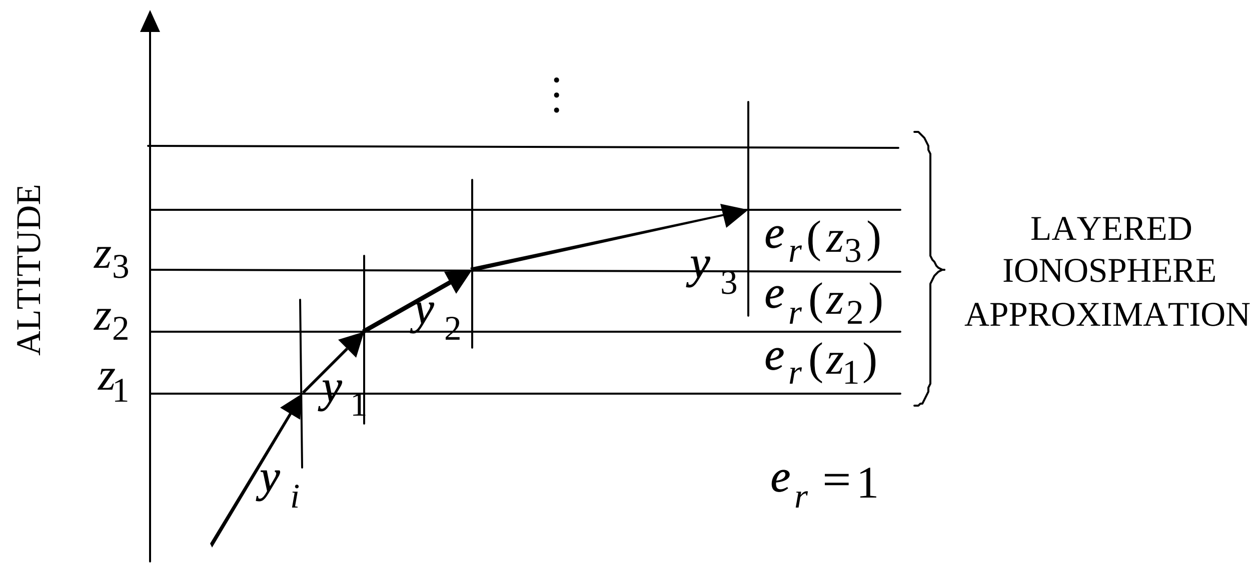
Ionospheric Radiowave Propagation (1)

If the wave's frequency and angle of incidence on the ionosphere are chosen correctly, the wave will curve back to the surface, allowing for very long distance communication. For oblique incidence, at a point of the ionosphere where the critical frequency is f_c, the ionosphere can reflect waves of higher frequencies than the critical one. When the wave is incident from a non-normal direction, the reflection appears to occur at a virtual reflection point, h', that depends on the frequency and angle of incidence.



Ionospheric Radiowave Propagation (2)

To predict the bending of the ray we use a layered approximation to the ionosphere just as we did for the troposphere.



Snell's law applies at each layer boundary

$$\sin y_i = \sin(y_1)\sqrt{e_r(z_1)} = \cdots$$

The ray is turned back when $y(z) = p/2$, or $\sin y_i = \sqrt{e_r(z)}$

Ionospheric Radiowave Propagation (3)

Note that:

1. For constant y_i, N_e must increase with frequency if the ray is to return to Earth (because e_r decreases with w).
2. Similarly, for a given maximum N_e ($N_{e\max}$), the maximum value of y_i that results in the ray returning to Earth increases with increasing w.

<u>There is an upper limit on frequency that will result in the wave being returned back to Earth</u>. Given $N_{e\max}$ the required relationship between y_i and f can be obtained

$$\sin y_i = \sqrt{e_r(z)}$$

$$\sin^2 y_i = 1 - \frac{w_p^2}{w^2}$$

$$1 - \cos^2 y_i = 1 - \frac{81 N_{e\max}}{f^2}$$

$$N_{e\max} = \frac{f^2 \cos^2 y_i}{81} \quad \Rightarrow \quad f_{\max} = \sqrt{\frac{81 N_{e\max}}{\cos^2 y_i}}$$

Ionospheric Radiowave Propagation (4)

Examples:

1. $y_i = 45^\circ$, $N_{e\max} = 2\times 10^{10}\ /\ \mathrm{m}^3$: $f_{\max} = \sqrt{(81)(2\times 10^{10})/(0.707)^2} = 1.8\ \mathrm{MHz}$
2. $y_i = 60^\circ$, $N_{e\max} = 2\times 10^{10}\ /\ \mathrm{m}^3$: $f_{\max} = \sqrt{(81)(2\times 10^{10})/(0.5)^2} = 2.5\ \mathrm{MHz}$

The value of f that makes $e_r = 0$ for a given value of $N_{e\max}$ is the critical frequency defined earlier:

$$f_c = 9\sqrt{N_{e\max}}$$

Use the $N_{e\max}$ expression from the previous page and solve for f

$$f = 9\sqrt{N_{e\max}}\sec y_i = f_c \sec y_i$$

This is called the <u>secant law</u> or <u>Martyn's law</u>. When $\sec y_i$ has its maximum value (maximum angle of incidence on the ionosphere), the frequency is called the <u>maximum usable frequency</u> (MUF). A typical value is less than 40 MHz. It can drop as low as 25 MHz during periods of low solar activity. The <u>optimum usable frequency</u>[1] (OUF) is 50% to 80% of the MUF. Data for the MUF is available. This is only correct for horizontally stratified media.

[1] Sometimes called the optimum working frequency (OWF).

Maximum Usable Frequency

Shown below is the MUF in wintertime for different skip distances. The MUF is lower in the summertime.

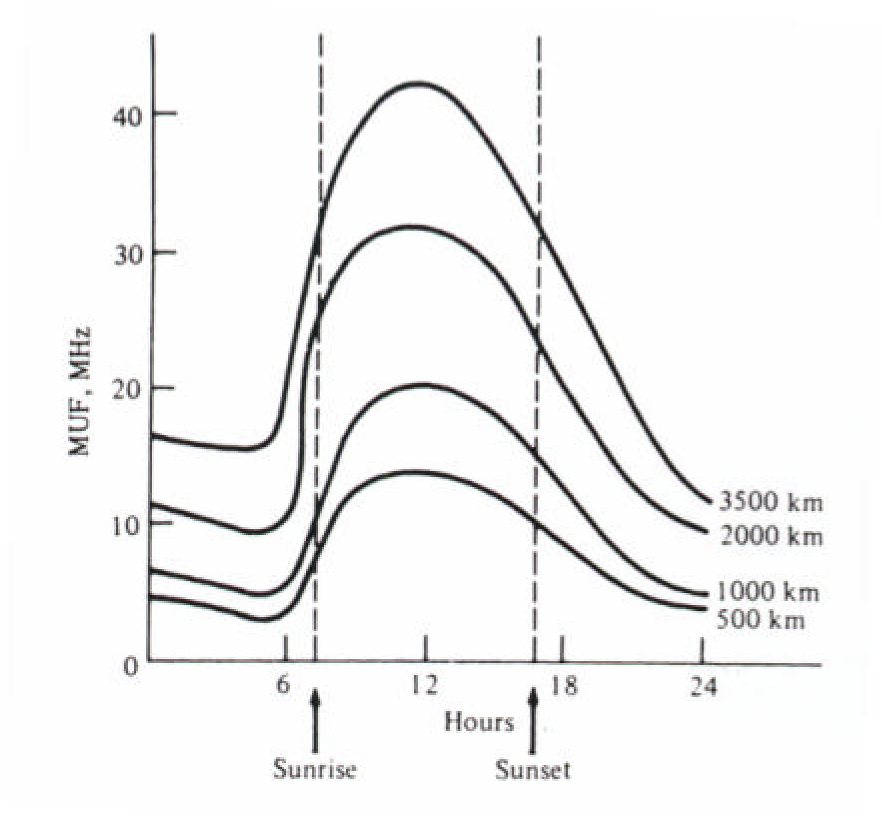
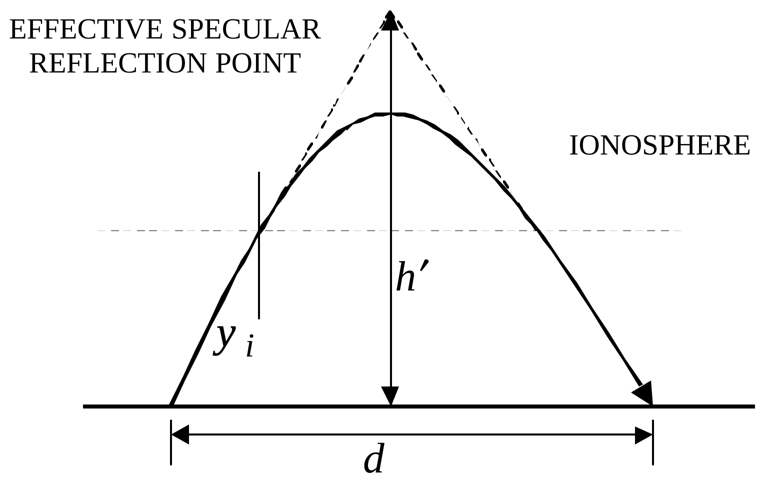


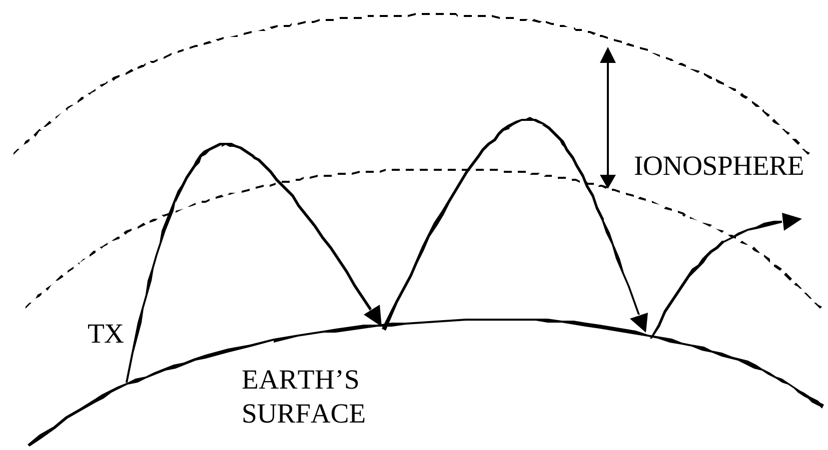
Fig. 6.43 in R. E. Collin, *Antennas and Radiowave Propagation*, McGraw-Hill, 1985

Ionospheric Radiowave Propagation (5)

Multiple hops allow for very long range communication links (transcontinental). Using a simple flat Earth model, the virtual height (h'), incidence angle (y_i), and skip distance (d) are related by $\tan y_i = \dfrac{d}{2h'}$. This implies that the wave is launched well above the horizon. However, if a spherical Earth model is used and the wave is launched on the horizon then $d = 2\sqrt{2R_e' h'}$.



Single ionospheric hop
(flat Earth)



Multiple ionospheric hops
(curved Earth)

Ionospheric Radiowave Propagation (6)

Approximate virtual heights for layers of the ionosphere

Layer	Range for h' (km)
F_2	250 to 400 (day)
F_1	200 to 250 (day)
F	300 (night)
E	110

Example: Based on geometry, a rule of thumb for the maximum incidence angle on the ionosphere is about 74°. The MUF is

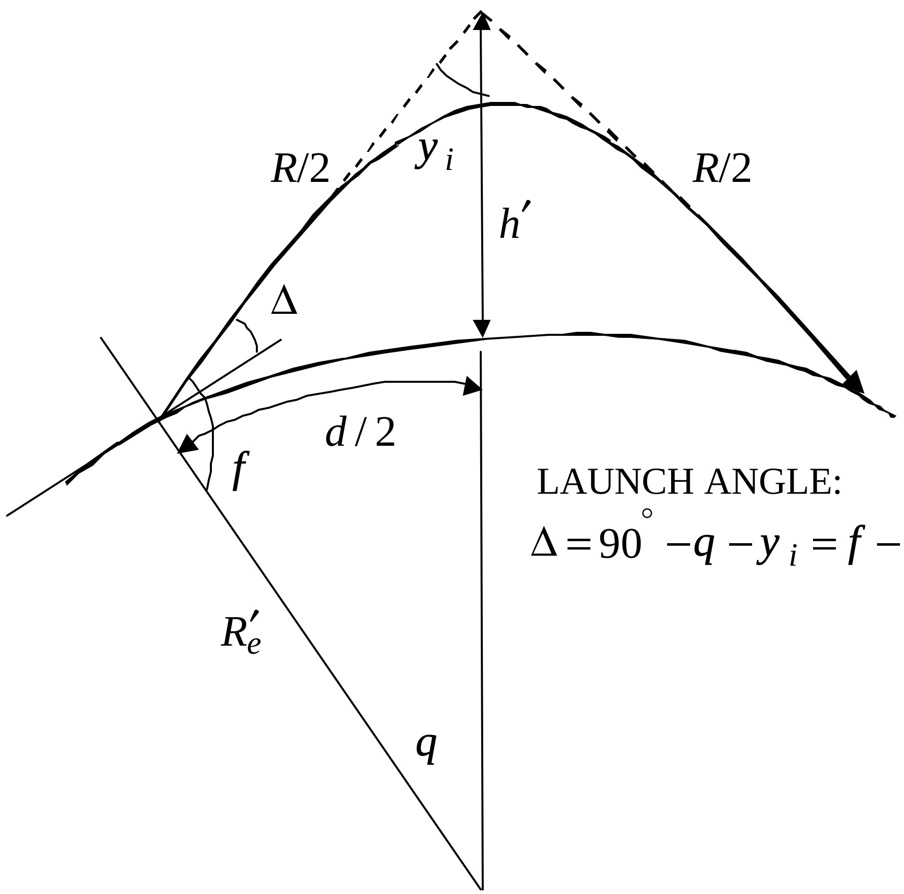
$$\mathrm{MUF} = f_c \sec(74^\circ) = 3.6 f_c$$

For $N_{e\,\max} = 10^{12}\ /\mathrm{m}^3$, $f_c \approx 9$ MHz and the $\mathrm{MUF} = 32.4$ MHz. For reflection from the F_2 layer, $h' \approx 300$ km. The maximum skip distance will be about

$$d_{\max} \approx 2\sqrt{2R_e' h'} = 2\sqrt{2(8500\times10^3)(300\times10^3)} = 4516\ \text{km}$$

Ionospheric Radiowave Propagation (7)

For a curved Earth, using the law of sines for a triangle $\dfrac{1 + h'/R_e' - \cos q}{\sin q} = \dfrac{1}{\tan y_i}$



LAUNCH ANGLE:
$\Delta = 90^\circ - q - y_i = f - 90^\circ$

where

$$q = \frac{d}{2R_e'}$$

and the launch angle (antenna pointing angle above the horizon) is

$$\Delta = f - 90^\circ = 90^\circ - q - y_i$$

The great circle path via the reflection point is R, which can be obtained from

$$R = \frac{2R_e' \sin q}{\sin y_i}$$

Ionospheric Radiowave Propagation (8)

Example: Ohio to Europe skip (4200 miles = 6760 km). Can it be done in one hop?

To estimate the hop, assume that the antenna is pointed on the horizon. The virtual height required for the total distance is

$$d/2 = R_e'\theta \quad \rightarrow \quad \theta = d/(2R_e') = 0.3976 \text{ rad} = 22.8 \text{ degrees}$$
$$(R_e' + h')\cos\theta = R_e' \quad \rightarrow \quad h' = R_e'/\cos\theta - R_e' = 720 \text{ km}$$

This is above the F layer and therefore two skips must be used. Each skip will be half of the total distance:. Repeating the calculation for $d/2 = 1690$ km gives

$$\theta = d/(2R_e') = 0.1988 \text{ rad} = 11.39 \text{ degrees}$$
$$h' = R_e'/\cos\theta - R_e' = 171 \text{ km}$$

This value lies somewhere in the F layer. We will use 300 km (a more typical value) in computing the launch angle. That is, still keep $d/2 = 1690$ km and $\theta = 11.39$ degrees, but point the antenna above the horizon to the virtual reflection point at 300 km

$$\tan\psi_i = \sin(11.39^\circ)\left[1 + \frac{300}{8500} - \cos(11.39^\circ)\right]^{-1} \quad \rightarrow \quad \psi_i = 74.4^\circ$$

Ionospheric Radiowave Propagation (9)

The actual launch angle required (the angle that the antenna beam should be pointed above the horizon) is

$$\text{launch angle, } \Delta = 90^\circ - q - y_i = 90^\circ - 11.39^\circ - 74.4^\circ = 4.21^\circ$$

The electron density at this height (see chart, p.3) is $N_{e\max} \approx 5 \times 10^{11} / \mathrm{m}^3$ which corresponds to the critical frequency

$$f_c \approx 9\sqrt{N_{e\max}} = 6.36 \text{ MHz}$$

and a MUF of

$$\text{MUF} \approx 6.36 \sec 74.4^\circ = 23.7 \text{ MHz}$$

Operation in the international short wave 16-m band would work. This example is oversimplified in that more detailed knowledge of the state of the ionosphere would be necessary: time of day, time of year, time within the solar cycle, etc. These data are available from published charts.

Ionospheric Radiowave Propagation (10)

Generally, to predict the received signal a modified Friis equation is used:

$$P_r = \frac{P_t G_t G_r}{(4pR / l)^2} L_x L_a$$

where the losses, in dB, are negative:

$L_x = L_{\text{pol}} + L_{\text{refl}} - G_{\text{iono}}$

L_{refl} = reflection loss if there are multiple hops

L_{pol} = polarization loss due to Faraday rotation and earth reflections

G_{iono} = gain due to focussing by the curvature of the ionosphere

L_a = absorption loss

R = great circle path via the virtual reflection point

Example: For $P_t = 30$ dBW, f = 10 MHz, $G_t = G_r = 10$ dB, d = 2000 km, $h' = 300$ km, $L_x = 9.5$ dB and $L_a = 30$ dB (data obtained from charts).

From geometry compute: $y_i = 70.3^\circ$, R = 2117.8 km, and thus $P_r = -108.5$ dBw

Magneto-ionic Medium (1)

Thus far the Earth's magnetic field has been ignored. An electron with velocity $\vec{u}$ moving in a medium with a magnetic field $\vec{B}_o = m_o \vec{H}_o$ experiences a force $\vec{F}_m = -e\vec{u} \times \vec{B}_o$.
Assume a plane wave that is propagating in the z direction ($\vec{E}, \vec{H} \sim e^{-gz}$). From Maxwell's equations

$$\nabla \times \vec{H} = jw\vec{D} = jw\left[e_o \vec{E} + \vec{P}\right] \quad \rightarrow \quad \begin{cases} gH_y = jwE_x\left(e_o + P_x / E_x\right) & (1) \\ gH_x = jwE_y\left(e_o + P_y / E_y\right) & (2) \\ \quad 0 = jw\left(e_o E_z + P_z\right) = jwD_z & (3) \end{cases}$$

$$\nabla \times \vec{E} = -jwm_o \vec{H} \quad \rightarrow \quad \begin{cases} gE_y = -jwm_o H_x & (4) \\ gE_x = jwm_o H_y & (5) \\ \quad 0 = -jwm_o H_z & (6) \end{cases}$$

These equations show that the plane wave will be transverse only with respect to the $\vec{B}$, $\vec{D}$ and $\vec{H}$ vectors, which is different from the isotropic case. The characteristic impedance of the medium is

$$h = E_x / H_y = -E_y / H_x = jwm_o / g \,.$$

Magneto-ionic Medium (2)

Solve (5) for H_y and use the result in (1): $g^2 = -w^2 m_o e_o \underbrace{\left(1 + \frac{P_x}{e_o E_x}\right)}_{n^2}$. A similar process starting with equation (4) would lead to $n^2 = 1 + \frac{P_y}{e_o E_y}$. Equating this to the expression for n^2 above gives the polarization ratio of the wave, $R_{\text{pol}} \equiv \frac{P_x}{P_y} = \frac{E_x}{E_y}$.

Rewrite the equation of motion for an average electron when both electric and magnetic fields are present

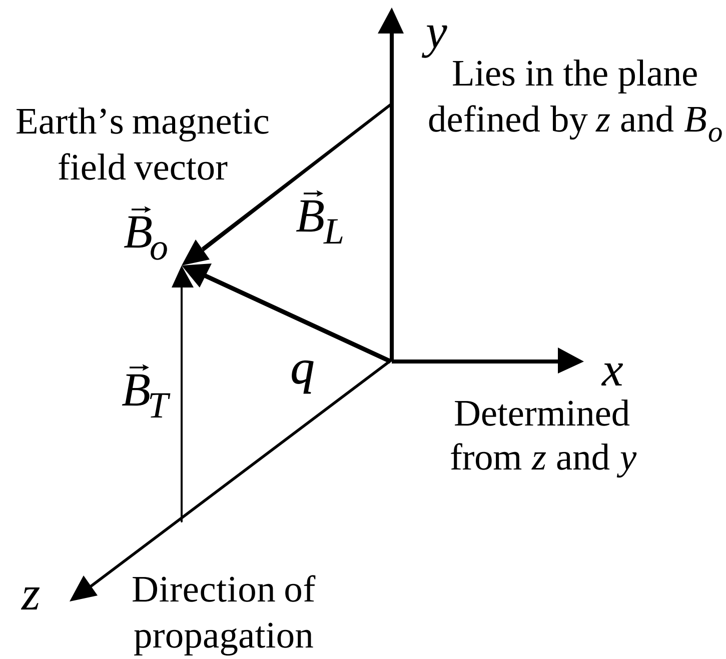
$$m\ddot{\vec{r}} + nm\dot{\vec{r}} = -e\left(\vec{E} + \dot{\vec{r}} \times \vec{B}_o\right)$$

There is also a term in the parenthesis for the magnetic field of the wave, but it turns out to be negligible. We now define the longitudinal and transverse components, which are parallel and perpendicular to the direction of propagation, respectively.

Magneto-ionic Medium (3)

Definition of the coordinate system: *y* lies in the plane defined by $B_o (= m_o H_o)$ and *z*

$$H_L = H_z = H_o \cos q \quad \text{(longitudin al)}$$
$$H_T = H_y = H_o \sin q \quad \text{(transverse)}$$



Note the direction of *x* is chosen to form a right-handed system (RHS). Any combination of propagation direction and magnetic field can be handled with this convention, as long as they are not parallel. If they are parallel any combination of *x* and *y* that forms a RHS is acceptable.

In this coordinate system the equation of motion can be written as three scalar equations, which for the *x* component is

$$m\ddot{x} + nm\dot{x} = -e\left(E_x + \dot{y} m_o H_L - \dot{z} m_o H_T\right)$$

Magneto-ionic Medium (4)

For time-harmonic displacements, phasors can be used and the derivatives in time give a multiplicative factor jw

$$-w^2 mx + jw n mx = -e\left(E_x + jwym_o H_L - jwzm_o H_T\right)$$

The polarization is caused by the electron displacements in x, y, and z

$$x = \frac{P_x}{-eN_e}, \quad y = \frac{P_y}{-eN_e}, \quad z = \frac{P_z}{-eN_e}$$

Substituting: $$\frac{w^2 mP_x}{eN_e} - \frac{jw n mP_x}{eN_e} = -eE_x - \frac{jwm_o P_y H_L}{eN_e} + \frac{jwm_o P_z H_T}{eN_e}$$

Multiply by e_o / m and define the longitudinal and transverse gyro-frequencies[1]:

$$w_L \equiv m_o e H_L / m = w_B \cos q$$
$$w_T \equiv m_o e H_T / m = w_B \sin q\ .$$

[1] In general, the gyro-frequency (also called the cyclotron frequency) of an electron in a magnetic field is defined as $w_B \equiv eB_o / m$ or $w_H \equiv em_o H_o / m$.

Magneto-ionic Medium (5)

For convenience we define the frequency ratios

$$X = \left(\frac{w_p}{w}\right)^2,\ Y_L = \frac{w_L}{w},\ Y_T = \frac{w_T}{w},\ \text{and}\ Z = \frac{n}{w}.$$

Recall that the plasma frequency is $w_p = \sqrt{\dfrac{N_e e^2}{m e_o}}$. With this notation

$$e_o X E_x = -P_x(1 - jZ) - jY_L P_y + jY_T P_z$$

Going back to the *y* and *z* components of the equation of motion, and treating them the same, gives two more equations

$$e_o X E_y = -P_y(1 - jZ) + jY_L P_x$$
$$e_o X E_z = -P_z(1 - jZ) - jY_T P_x$$

These are the equations needed to find *n* in terms of the components of $\vec{E}$, and $\vec{P}$. Arbitrarily we choose to solve for P_y and E_y, eliminating P_x, E_x, P_z, E_z.

Magneto-ionic Medium (6)

With several substitutions and a fair amount of algebra, two equations are obtained:

$$e_o X R_{\mathrm{pol}} E_y = -P_y R_{\mathrm{pol}}(1 - jZ) - jY_L P_y + Y_T^2 R_{\mathrm{pol}} P_y / (1 - X - jZ)$$
$$e_o X R_{\mathrm{pol}} E_y = -P_y R_{\mathrm{pol}}(1 - jZ) + jY_L P_y R_{\mathrm{pol}}^2$$

Equate the right hand sides: $-jY_L + Y_T^2 R_{\mathrm{pol}} / (1 - X - jZ) = jY_L R_{\mathrm{pol}}^2$

Solving this quadratic equation gives:

$$R_{\mathrm{pol}} = -\frac{j}{Y_L}\left\{\frac{Y_T^2}{2(1 - X - jZ)} \mp \left[\frac{Y_T^4}{4(1 - X - jZ)^2} + Y_L^2\right]^{1/2}\right\}$$

Not all plane waves can pass through the medium. To remain plane waves they must have one of two polarizations that are characteristic of the medium. If this equation is satisfied, both equations at the top of the page give the same solution:

$$P_y / E_y = -\frac{e_o X}{1 - jZ - jY_L R_{\mathrm{pol}}}$$

Magneto-ionic Medium (7)

Use this ratio to find the index of refraction: $n^2 = 1 - X / \left[1 - jZ - jY_L R_{\text{pol}}\right]$ and with R_{pol} from the quadratic equation yields the Appleton-Hartree formula

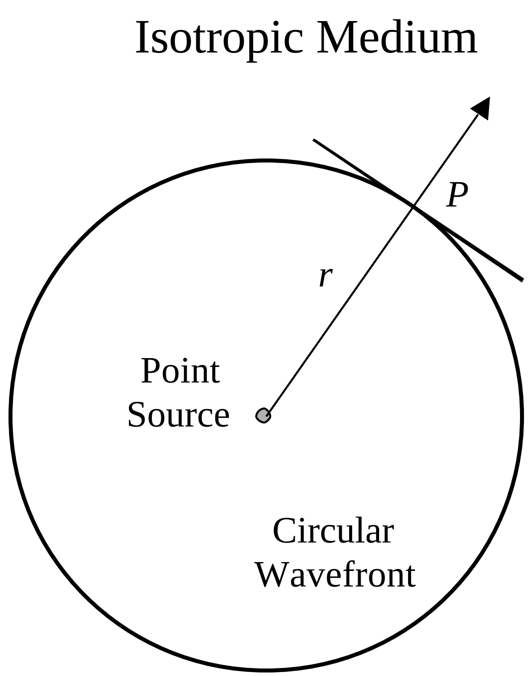
$$e_r = n^2 = (n' - jn'')^2 = 1 - \frac{X}{1 - jZ - \dfrac{Y_T^2}{2(1 - X - jZ)} \pm \left\{ \dfrac{Y_T^4}{4(1 - X - jZ)^2} + Y_L^2 \right\}^{1/2}}$$

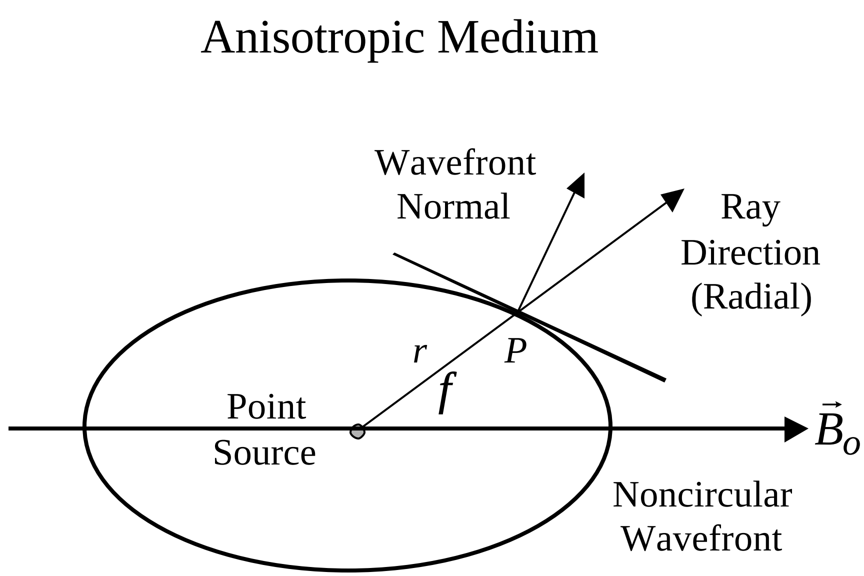
Physical interpretation: The Earth's magnetic field causes the ionosphere to be anisotropic. There are two modes of propagation, each with a particular polarization, that depends entirely on the properties of the medium. The phase velocities of the two modes, from the two values of n above, are different, and when they recombine they have different phase relationships. The solution with the positive sign is called the ordinary wave; that with the negative sign is the extraordinary wave. When the frequency is greater than about 1 MHz, which is always the case for systems that transmit through the ionosphere, the wave can be considered solely longitudinal (H_T, w_T and Y_T are zero) and therefore,

$$n^2 \approx 1 - \frac{w_p^2}{w(w \pm w_L - jn)}$$ is sufficiently accurate for most calculations.

Anisotropic Media

In an isotropic medium the phase velocity at any given point is independent of the direction of propagation. For an anisotropic medium, the direction of phase propagation (i.e., movement of the equiphase planes) differs, in general, from that of energy propagation.





Phase Velocity and Time Delay (1)

The time delay between a transmitter at P_1 and receiver at P_2 is: $t_d = \frac{1}{c}\int_{P_1}^{P_2} \frac{1}{n'(s)}\, ds$

The angular (radian) phase path length is: $\Phi = \frac{w}{c}\int_{P_1}^{P_2} n'(s)\, ds$.

Let the distance between point P_1 and P_2 be ℓ. As a check, for a homogeneous medium

$$\Phi = \frac{w}{c}\ell\sqrt{e_r} = b\ell$$

The phase velocity is $u_p = \frac{w}{b}$, which can be determined once b is known. The group velocity is $u_g = \frac{dw}{db}$. The velocity of energy propagation is generally taken as the group velocity[1]. It is always true that $u_p u_g = c^2$.

1 This is the velocity of propagation of a packet of frequencies centered about a carrier frequency. For simple amplitude modulated (AM) waveforms, it is the velocity of propagation of the envelope, whereas the phase velocity is the velocity of propagation of the carrier.

Phase Velocity and Time Delay (2)

Neglect the magnetic field and consider the case where $Z^2 << 1$ and $w_p^2 << w^2$, which is typical for links that penetrate the ionosphere. Therefore

$$n \approx n' = \sqrt{1 - X^2} \approx 1 - \frac{1}{2}\left(\frac{w_p}{w}\right)^2 = 1 - \frac{1}{2}\left(\frac{N_e e^2}{m e_o w^2}\right)$$

The phase of the path is

$$\Phi = -\frac{w}{c}\int_{P_1}^{P_2}\left\{1 - \frac{1}{2}\left(\frac{N_e e^2}{m e_o w^2}\right)\right\} ds = -\frac{w}{c}\ell + \frac{1}{2}\left(\frac{e^2}{m e_o w\, c}\right)\underbrace{\int_{P_1}^{P_2} N_e(s) ds}_{N_T}$$

The integral N_T (/m^2) represents the <u>total electron content</u> (TEC) along the path from P_1 to P_2. The second term is the phase error due to ionization. A typical value for a vertical path at 100 MHz is 3×10^{17} electrons/m^2, which gives

$$\frac{1}{4p}\frac{\left(3\times10^{17}\right)\left(1.6\times10^{-19}\right)^2}{\left(9\times10^{-31}\right)\left(8.85\times10^{-12}\right)\left(3\times10^{8}\right)\left(100\times10^{6}\right)} \approx 2526 \text{ rad}$$

Phase Velocity and Time Delay (3)

The phase rotation of the ordinary wave is found to be

$$\Phi_{\mathrm{o}} = -k_o \ell + \frac{N_T}{2 - Y_T^2 - \sqrt{Y_T^4 + 4Y_L^2}} \left(\frac{e^2}{m e_o w c} \right) \quad \text{Note}: \frac{e^2}{m e_o c} = 1.06 \times 10^{-5}$$

For the extraordinary wave

$$\Phi_{\mathrm{x}} = -k_o \ell + \frac{N_T}{2 - Y_T^2 + \sqrt{Y_T^4 + 4Y_L^2}} \left(\frac{e^2}{m e_o w c} \right)$$

<u>Faraday rotation</u> is the rotation of the phase angle of a linearly polarized plane wave.

$$\Omega = \frac{1}{2}(\Phi_{\mathrm{o}} - \Phi_{\mathrm{x}}) = \frac{1}{4}\left(\frac{e^2}{m e_o w c} \right) N_T \frac{\sqrt{Y_T^4 + 4Y_L^2}}{1 - Y_T^2 - Y_L^2} \approx \frac{1}{4}\left(\frac{e^2}{m e_o w c} \right) N_T \sqrt{Y_T^4 + 4Y_L^2}$$

where the approximation assumes that the wave frequency is much larger than the gyro frequency. With the same assumptions, the time delay is

$$t_d \approx \frac{1}{c} \int_{P_1}^{P_2} \left\{ 1 + \frac{1}{2}\left(\frac{N_e e^2}{m e_o w^2} \right) \right\} ds = \frac{\ell}{c} + \frac{1}{2}\left(\frac{e^2}{m e_o w^2 c} \right) N_T$$

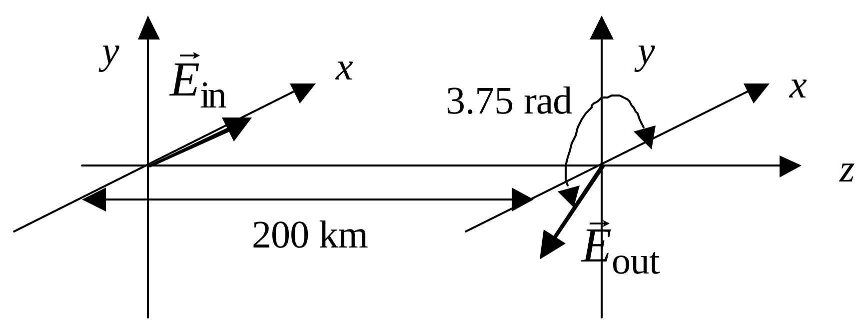
With a magnetic field present the equations are more complex.

Example

Find the Faraday rotation of a 135 MHz plane wave propagating a distance of 200 km through the ionosphere. The electron density is constant over the path and equal to 5×10^{10} /m^3. The gyro-frequency is 8×10^6 Hz. Assume collisions can be neglected and that the propagation is longitudinal.

The TEC is $N_T = \int_0^{\ell} N_e(s)ds = \ell N_e = 200\times10^3\left(5\times10^{10}\right) = 10^{16}$ /m^2. For longitudinal propagation and no collisions $Y_T = Z = 0$ and $Y_L = w_B / w$

$$\begin{aligned}\Omega &\approx \frac{1}{4}\left(\frac{e^2}{m e_o w c}\right) N_T\, 2\left(\frac{w_B}{w}\right)\\ &\approx \frac{\left(1.06\times10^{-5}\right)\left(10^{16}\right)\left(2p\right)\left(8\times10^6\right)}{2\left(2p\,135\times10^6\right)^2}\\ &\approx 3.75 \text{ radians}\end{aligned}$$



Absorption

Once the index of refraction is determined for the medium, the attenuation constant can be obtained from

$$g = a + jb = j\frac{w}{c}n.$$

For simplicity, the effect of the magnetic field is ignored in this discussion ($Y_T = Y_L = 0$) so that

$$n^2 = (n' - jn'')^2 = \left(1 - \frac{X}{1+Z^2} - j\frac{XZ}{1+Z^2}\right)$$

yielding $a = \frac{w}{c}n'' = \frac{w}{c}\frac{1}{2n'}\frac{XZ}{1+Z^2}$. In general, a is not constant along a path through the ionosphere, because the electron density, and hence X and n' are changing along the path. The collision frequency, if not zero, can also change with location in the ionosphere. Thus, the loss due to absorption from P_1 to P_2 should be computed from

$$L_{\text{attn}} = \exp\left(-\int_{P_1}^{P_2} a(s)\,ds\right)$$

where s is the distance along the path.

Example

Find the attenuation (in dB/km) in the ionosphere for $f = 30$ MHz, $n = 10^7$ /s and $N_e = 10^8$ /m^3.

Using the formula from the previous page with $Z = \left(\dfrac{n}{w}\right) = \left(\dfrac{10^7}{2p\,30\times10^6}\right) = 0.0531$

and $X = \left(\dfrac{w_p}{w}\right)^2 = \left(\dfrac{56.34\sqrt{10^8}}{2p\,30\times10^6}\right)^2 = 8.9\times10^{-6}$, $e_r' = 1 - \dfrac{N_e e^2}{e_o m\left(n^2 + w^2\right)} \approx 1.$ Substituting gives:

$$a = \frac{w}{c}\frac{1}{2\sqrt{e_r'}}\frac{XZ}{1+Z^2} = 1.48\times10^{-7} \rightarrow -20\log(e^{-1000a}) = 0.0013 \text{ dB/km.}$$

Alternatively, we could use the basic equation for attenuation

$$a = w\sqrt{\frac{m_o e_o e_r'}{2}\left(\sqrt{1+\left(\frac{s}{w e_o e_r'}\right)^2} - 1\right)} \quad \text{where} \quad s = \frac{N_e e^2 n}{m\left(n^2 + w^2\right)}.$$

For $N_e = 10^{11}$ the attenuation increases to 1.3 dB/km.